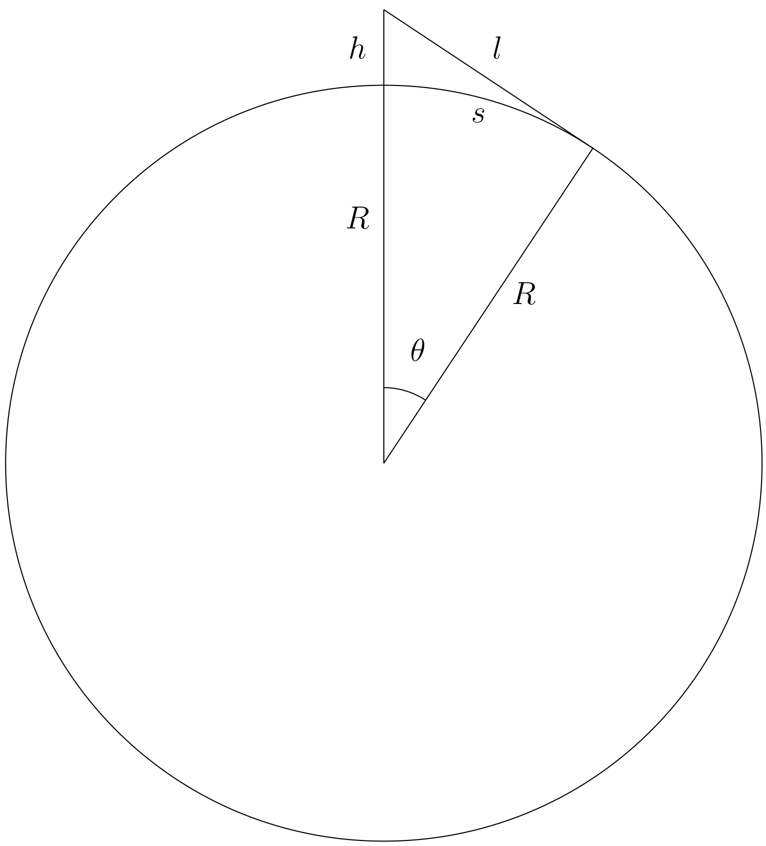
How far away is the horizon?

When we stand on the shore, with our feet just at sea level, and look out to sea, we may ask: how far away is the horizon?

We start by making the simplifying assumption that the earth is a sphere with radius R. Then the line from our eye to the horizon is tangent to a circle of radius R (representing the earth) as shown:



Here h is our height (or the height of our eye) above sea level, R is the radius of the earth, l is the length of the line connecting our eye directly to the horizon, and s is the arc length, the distance along the surface of the earth to the horizon. The angle θ is the angle made between our position and the horizon.

Since the triangle in the figure is a right triangle, we may use the Pythagoren theorem to find relationships between the variables.

Specifically,

$$l = \sqrt{(R+h)^2 - R^2} = \sqrt{2Rh + h^2}.$$

If we make the assumption that l should be approximately the same as s (since h will be very small compared to R), then we can say

$$s \approx l = \sqrt{2Rh + h^2} \approx \sqrt{2Rh} = k\sqrt{h}$$

for a constant k. Here we have made the additional simplification that $2Rh + h^2 \approx 2Rh$ since h^2 will be very small compared to $2Rh$.

Working out this k value, we can use $R = 3959$ miles so that, with h in feet, we have

$$\sqrt{2Rh} = \sqrt{41807040h} = \frac{\sqrt{41807040}}{5280}\sqrt{h} \text{ miles } \approx 1.22459\sqrt{h} \text{ miles }.$$

For example, if our eye is 5 feet about sea level, then the horizon distance is about

$$1.22459\sqrt{5} = 2.738 \text{ miles away.}$$

To check our calculations, we can work out an "exact" formula and compare.
The angle θ can be found as

$$\theta = \cos^{-1} \frac{R}{R+h}$$

so that

$$s = R\theta = R\cos^{-1} \frac{R}{R+h}.$$

This expression is, perhaps surprisingly, extremely close to our previous, simpler expression. Here's a table of a few values (h in feet, other distances in miles).

h	$1.22459\sqrt{h}$	$s = R\cos^{-1} \frac{R}{R+h}$	percentage difference
5	2.7383	2.7378	-0.000008172203854%
10	3.8725	3.8728	0.000001794221200%
20	5.4765	5.4763	0.00002172705563%
50	8.6592	8.6590	0.00008152553997%
100	12.246	12.246	0.0001811895939%
500	27.383	27.382	0.0009784981053%
1000	38.725	38.724	0.001975123947%
10000	122.459	122.434	0.01991252800%
100000	387.249	386.478	0.1990930664%

As you can see, the simpler formula yields incredible accuracy to the complex one, at least to a height of 100000 feet! Notice that at 100000 feet, the angle θ is still only 5.5932 degrees. So, for all practical purposes, we can estimate the distance to the horizon using $1.22459\sqrt{h}$ (or perhaps even more simply as just $1.2\sqrt{h}$), which gives the distance in miles when h is in feet. The metric version is

$$s \approx 3.56972\sqrt{h} \approx 3.6\sqrt{h}$$

in kilometers, with h in meters.

Source: https://sites.math.washington.edu/~conroy/m120-general/horizon.pdf (Accessed 21/12/2020)